Nonlinear friction effect
in air-pressure forcing of the flow
through the Strait of Gibraltar

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ABSTRACT
The simple analytical two-strait and two-basin model built by (Candela et al., 1989), which well describes the barotropic flow through the Strait of Gibraltar forced by homogeneous fields of oscillating atmospheric pressure over the Eastern and Western Mediterranean, was improved by allowing the coefficient of linear bottom friction $\lambda$ to be frequency-dependent. The coefficient $\lambda$ is here proportional to the magnitude of the volume flow through the Strait of Gibraltar, and therefore increases with the frequency at very low frequencies, starting from zero. After reaching the extreme, which was tuned to be close to the value of frequency-independent $\lambda$, the coefficient slowly decreases with frequency. Both approaches, with frequency-dependent and independent $\lambda$, give almost the same frequency dependence for all variables, therefore the disagreement between model values and cross-spectral analysis remains at high frequencies. Using the values of cross-section dimensions proposed by Candela et al. the estimates of the drag coefficient are by about two orders of magnitude too high in both approaches. If the geostrophic control of the flow through the Strait of Gibraltar plays a role at low frequency range also when friction is included, then its upper frequency limit is estimated to be $\sim 4 \times 10^{-3}$ cph.


RÉSUMÉ
Frottement non linéaire dû au forçage de la pression de l'air sur le flux dans le détroit de Gibraltar

Le modèle analytique simple du détroit de Gibraltar proposé par Candela et al. (1989), avec deux détroits et deux bassins, décrit correctement le flux barotrope forcé par des champs homogènes de pression atmosphérique qui oscillent entre l'est et l'ouest de la Méditerranée. Ce modèle est amélioré lorsque le coefficient linéaire de frottement sur le fond varie avec la fréquence. Ici le coefficient est proportionnel au flux traversant le détroit de Gibraltar ; il augmente donc avec la fréquence aux très basses fréquences, en partant de zéro. Après avoir atteint un maximum proche de la valeur indépendante de la fréquence, le coefficient décroît légèrement lorsque la fréquence continue à augmenter. Les deux approches, avec coefficient variable ou indépendant de la fréquence, donnent pour tous les autres paramètres la même loi de variation avec la fréquence ; par conséquent, la différence entre les valeurs du modèle et l'analyse spectrale persiste aux fréquences élevées. Les valeurs des dimensions de la section proposées par Candela et al. conduisent à des estimations du coefficient de frottement qui sont trop élevées d'environ deux ordres de grandeur dans les deux approches. Si le contrôle géostrophique du flux à travers le détroit de Gibraltar joue un rôle dans la gamme des basses fréquences, même lorsque le frottement est pris en compte, la limite supérieure de la fréquence est estimée de l'ordre de $4 \times 10^{-3}$ cycles par heure.

INTRODUCTION

Nonlinear friction has mainly been invoked in the tidal analysis of shallow seas. In tidal spectroscopy in 1D channels closed at one end, odd harmonics due to bottom friction were evaluated approximately by Gallagher and Munk (1971) and a perturbation method was applied for them by Kabbaj and Le Provost (1980). They also compared both analytical approximations with numerical solutions of equations of motion. Iterative procedure, the renormalization, was used by Zimmerman (1982) for a quadratically-damped forced oscillator, and for a co-oscillated basin connected with an open sea through a narrow strait with nonlinear friction, which also behaves like an oscillator (Zimmerman, 1991; 1992).

All the above-mentioned methods concerning nonlinear friction were applied to problems, which reduce to an oscillator of some kind. Here, an attempt has been made to make the nonlinear correction of the model built by Candela, Winant and Bryden (1989), hereafter referred as CWB, which seems to the author to be one of the most accomplished of the "two-strait, two-basin" series of models, developed for describing the barotropic subinertial flows through straits of the Mediterranean Sea forced by an oscillated atmospheric pressure field. From the linearization of the bottom friction stress it follows that the coefficient of the linearized friction stress is proportional to the amplitude of velocity of the rectilinear flow. Therefore the coefficient involved in linearized friction stress is frequency-dependent. This last relaxation of the coefficient out from the constant value was also suggested by CWB in their conclusions as a possible next step in the parametrization of the model.

For the rectilinear flow, quadratic bottom friction stress produces higher (odd) harmonics. From the Fourier analysis of bottom friction stress it follows that only the first six of them are reasonably significant. We shall require that the bottom stress of the rectilinear motion should be correctly approximated for the fundamental harmonic. This means also that the average energy loss within one cycle of rectilinear motion is the same in both approaches, with and without linearization (Bowden, 1983). This goal is also achieved with a linear approximation of the first term in the Fourier series of the bottom friction stress for the rectilinear flow. The well-known relation between friction parameters (Zimmerman, 1991) is then obtained

\[ \lambda = \frac{8C_dU}{3\pi H} \]  

where \( C_d \) is the drag coefficient and \( U \) is the amplitude of the vertically averaged velocity of the rectilinear flow of the water column of depth \( H \).

MODEL PRESENTATION

The simple model developed by CWB satisfactorily describes the barotropic volume flow through the Strait of Gibraltar, where the forcing agent is the first mode of EOF analysis of the atmospheric pressure field over the Mediterranean Sea. We shall adopt their model in our consideration of nonlinear friction. The conservations of volume in the first (W Mediterranean) and the second (E Mediterranean) basin are therefore given as

\[ A_1 \frac{\partial \eta_1}{\partial t} = q_{G1} - q_{S1} \]  
\[ A_2 \frac{\partial \eta_2}{\partial t} = q_S \]  

(2.1)  
(2.2)

where \( q_{G1} = u_1 A_G \), \( q_{S1} = u_2 A_S \) are the volume flows through the first (Gibraltar) strait, and the second (Sicily) strait, \( A_G \) and \( A_S \) are their cross-section areas, while \( \eta_1 \) and \( A_1 \) (\( \eta_2 \) and \( A_2 \)) are the uniform sea-level elevation and the surface area of the first (second) basin. The subsurface uniform pressure in each basin is written simply as

\[ p_1 = p_{1a} + \rho g \eta_1 \]  
\[ p_2 = p_{2a} + \rho g \eta_2 \]  

(2.3)  
(2.4)

where \( p_{1a} \) and \( p_{2a} \) are the spatially averaged atmospheric pressures over basin 1 and 2, respectively. The last two equations are the equations of motion through straits

\[ \frac{\partial q_{G1}}{\partial t} = -\frac{A_0 p_1}{\rho L_G} - \lambda q_{G1} \]  
\[ \frac{\partial q_S}{\partial t} = -\frac{A_0 (p_2 - p_1)}{\rho L_S} \]  

(2.5)  
(2.6)

In the equation of motion through the Strait of Sicily the pressure gradient was approximated with \( (p_2 - p_1)/L_S \), where \( L_S \) is the length of the strait. For this strait, following the same argument of CWB of six times larger cross-sectional area of Sicily than of Gibraltar, the water velocities and frictional effects should be small, therefore the frictional term was omitted as in the CWB model. Further clarification of this approximation will follow in the discussion. The pressure gradient along the Strait of Gibraltar was approximated with \( p_1/L_G \), where \( L_G \) is the length of the Strait of Gibraltar, meaning that isostatic pressure variations due to the atmosphere are ignored on the Atlantic side of Gibraltar. All dimensions of straits and basins used by CWB will also be used here. CWB analyzed observations of atmospheric pressure sampled every six hours from 64 coastal meteorological stations using the EOF method. They revealed that 65% of the variance accounts for the spatial structure of the first mode, which represents a homogeneous, standing pattern without sign change. In our case the only forcing agent at subinertial frequencies are the atmospheric pressure fields averaged over each of the basins. So we shall write them as

\[ p_{1a} = \Re \{ p_{1a} e^{i\omega t} \} \]  
\[ p_{2a} = \Re \{ p_{2a} e^{i\omega t} \} \]  

(2.7)

We shall again look for the solutions in the form of trigonometric functions

\[ \eta_{1,2} = \Re \{ Z_{1,2} e^{i\omega t} \} \]  
\[ p_{1,2} = \Re \{ P_{1,2} e^{i\omega t} \} \]  
\[ q_{G,S} = \Re \{ Q_{G,S} e^{i\omega t} \} \]  

(2.8)
Inserting them in (2.1) to (2.6), together with forcing functions (2.7), for which amplitudes \( P_{a1} \) and \( P_{a2} \) are real, we obtain coupled equations for the complex amplitudes

\[
i\omega A_{1}Z_1 = Q_0 - Q \tag{2.9}
\]

\[
i\omega A_{2}Z_2 = Q_s \tag{2.10}
\]

\[
P_1 = P_{a1} + \rho g Z_1 \tag{2.11}
\]

\[
P_2 = P_{a2} + \rho g Z_2 \tag{2.12}
\]

\[
\omega Q_0 = -\frac{A_0 P_1}{\rho L_s} - \lambda Q_0 \tag{2.13}
\]

\[
\omega Q_s = -\frac{A_0 (P_2 - P_1)}{\rho L_s} \tag{2.14}
\]

Only the real parts of all quantities written in a complex form are considered at the end of the derivation procedure. The above system of amplitude equations is linear if the coefficient \( \lambda \) is not a function of the flow through Gibraltar Strait, and was estimated by CWB from \( \lambda = C_D U_0 H_0 \) to be 1.27 x 10^-5 s^-1 where for the hydraulic depth of the Strait of Gibraltar \( H_0 = 120 \) m was used, the drag coefficient \( C_D = 3 \times 10^{-3} \) and the amplitude of velocity \( U_0 = 0.5 \) m/s. But, as will be pointed out later, four times larger value had to be used to adjust the model with the data of the flow through the Strait of Gibraltar. Here, according to (1.1), \( \lambda \) will be proportional to the amplitude of the volume flow \( Q_0 \), but since this is shifted in phase with regard to the atmospheric pressure fields, the modulus \( |Q_0| \) should be used

\[
\lambda = \frac{8C_D}{3\pi A_0 H_0} |Q_0| = C_D |Q_0| \tag{2.15}
\]

where \( C_D = 7.2 \times 10^{-12} \) m^-3 if we use \( A_0 = 2.95 \times 10^6 \) m^2, \( H_0 = 120 \) m, and \( C_D = 3 \times 10^{-3} \). Later, different values of \( C_D \) will be considered to obtain model results similar to results of CWB. Inserting \( Z_1 \) from (2.9) and \( P_1 \) from (2.11) into (2.13) we obtain the first relation between \( Q_0 \) and \( Q_s \)

\[
Q_0 \left[ 1 - \frac{g A_0}{L_s A_1 \omega^2 - i \lambda} \right] = -\frac{g A_0 Q_s}{A_1 \omega^2 L_s} + \frac{i A_0 P_{a1}}{\rho \omega L_s} \tag{2.16}
\]

For the second relation we put \( Z_2 \) from (2.12), \( P_2 \) from (2.14), and \( P_1 \) from (2.13) into (2.10)

\[
Q_s \left[ 1 - \frac{g A_2}{g A_0} \right] = \frac{L_s \omega A_0 Q_0 (\omega - i \lambda)}{g A_0} + \frac{i A_0 P_{a2}}{\rho g} \tag{2.17}
\]

Before solving the system of (2.16) and (2.17) it seems to be more appropriate to introduce another two constants

\[
\Omega^2 = \frac{g A_0}{L_s A_1} ; \quad \Omega_\Pi^2 = \frac{g A_1}{L_s A_2} \tag{2.18}
\]

For values of \( L_s = 6 \times 10^4 \) m, \( L_s = 10^5 \) m, \( A_0 = 2.951 \times 10^6 \) m^2, \( A_1 = 1.8926 \times 10^7 \) m^2, \( A_1 = 8.646 \times 10^{11} \) m^2, and \( A_2 = 1.6703 \times 10^{12} \) m^2 (Tab. 2 of CWB) they become: \( \Omega^2 = 0.558 \times 10^{-9} \) s^-2 and \( \Omega_\Pi^2 = 1.11 \times 10^{-9} \) s^-2. Considering them as angular frequencies, they correspond to the periods of ~2.2 days and ~3.1 days. We shall express the amplitudes of the atmospheric pressure fields in elevation units \( Z_{a1} = P_{a1}/(\rho g) \) and \( Z_{a2} = P_{a2}/(\rho g) \), where \( \rho \) is the density of the sea water. The resulting equation for \( Q_0 \) may be written in a similar form to that obtained by CWB

\[
Q_0 \left[ B + i\lambda C \right] = i\omega \left[ A_1 Z_{a1} (1 - \frac{\omega^2}{\Omega^2}) + A_2 Z_{a2} \right] \tag{2.19}
\]

with

\[
B = -1 + \omega^2 \left[ \frac{1}{\Omega^4} \left( 1 + \frac{A_2}{A_1} \right) \right] \frac{1}{\Omega_\Pi^4} \frac{1}{\Omega_\Pi^4} - \frac{\omega^4}{\Omega_\Pi^4} \tag{2.20}
\]

\[
C = \frac{\omega^2}{\Omega_\Pi^4} \left( \frac{\omega^2}{\Omega_\Pi^4} - 1 - \frac{A_2}{A_1} \right) \tag{2.21}
\]

where \( D_0 \) in CWB was replaced with \( \lambda \). Now, we shall multiply both sides of the above complex equation with their conjugate expressions, and we shall introduce \( \lambda \) from (2.15). The amplitude equation for \( |Q_0| \) finally follows as

\[
|Q_0|^2 (C_D)^2 + |Q_0|^2 B^2 - \mathcal{A}^2 = 0 \tag{2.22}
\]

where coefficient \( \mathcal{A} \) is

\[
\mathcal{A} = A_1 Z_{a1} (1 - \frac{\omega^2}{\Omega^2}) + A_2 Z_{a2} \tag{2.23}
\]

Coefficient \( C = 0 \) when \( \omega = \Omega_\Pi \frac{(1 + A_2/A_1)^{1/2}}{5.7 \times 10^5} \) s^-1 and the corresponding period is around 30.6 hours. We shall denote this particular angular frequency with \( \omega_0 \). The solution for \( |Q_0| \) for this angular frequency does not depend on the friction coefficient \( C_D \)

\[
|Q_0| = \frac{1}{\sqrt{\mathcal{B}}(i \mathcal{C})^2} \tag{2.24}
\]

For \( C \neq 0 \) the single positive solution of (2.19) is easily obtained as

\[
|Q_0| = \frac{\omega}{\sqrt{B^2 + (\lambda C)^2}} \tag{2.25}
\]

In the above expression the volume flow through the Strait of Gibraltar is a more complicated function of angular frequency than \( |Q_0| \) obtained by CWB for \( \lambda \) independent of \( \omega \), which follows directly from (2.19).

\[
|Q_0| = \frac{\omega |\mathcal{A}|}{\sqrt{B^2 + (\lambda C)^2}} \tag{2.26}
\]

Multiplication of (2.20) by a constant \( C_D \) gives us the dependence of \( \lambda \) on \( \omega \), according to (2.15), and also on the forcing amplitudes \( Z_{a1} \) and \( Z_{a2} \), which is a characteristic of nonlinear friction law.

**DISCUSSION-CONCLUSION**

The derivation of all unknowns for \( \lambda \) being a function of frequency, may start with \( Q_0 \) from (2.18), in which we insert \( \lambda = C_D |Q_0| \), with \( |Q_0| \) from (2.20). Plots of their abs-
Values of frequency-independent $\lambda$ used by CWB in the model are $5.17 \times 10^{-5}$ s$^{-1}$, $3.0 \times 10^{-5}$ s$^{-1}$, and $8.0 \times 10^{-5}$ s$^{-1}$. They have been also used here. The coefficient of linear friction increases with frequency (together with |$Q_{Ql}$|) almost linearly at very low frequencies, with a smaller rate for large values of $C_d$. After reaching the extreme, which is very close to the adequate value of $\lambda$ used by CWB, the coefficient slowly decreases with frequency, this time with a higher rate for large values of $C_d$. Bearing in mind the value for $A_G H_G = 3.54 \times 10^8$ m$^2$, used by CWB, this implies that values for the drag coefficient $C_d$ are to be 0.198, 0.065, and 0.462, respectively, which is certainly by about two orders of magnitude too high.

Let the frequency-dependent $\lambda$ be equal to the frequency-independent value, say $\lambda = 5.17 \times 10^{-5}$ s$^{-1}$. From Figure 1 we can easily obtain the two frequencies at which this happens as 0.0048 and 0.0113 cph. Also at these frequencies, gains $|Q_{Ql}|$ (Fig. 2) of both models exactly meet and are equal (solid lines) to 0.092 Sv (1 Sv = $10^6$ m$^3$ s$^{-1}$). Using this value for the volume flow in (2.15) $C_d = 0.23$ which is again by two orders of magnitude too high. A pronounced reduction of the “hydraulic” depth $H_G$ by about one order of magnitude would give us reasonable values for the drag coefficient for both models using different concepts of $\lambda$.

Nonetheless, both approaches give quite similar behaviour for the volume flow through the Strait of Gibraltar, and they both fail in describing the flow at frequencies higher than $8 \times 10^{-3}$ cph. At these frequencies the barotropic flow at Gibraltar, according to EOF analysis of CWB, is not related to the first EOF mode of atmospheric pressure, supposed to be in forcing function, over the Mediterranean. The coherence between atmospheric pressure and the flow through Gibraltar is small in this frequency range. The values of the cross-spectral analysis of the volume flow through the Strait of Gibraltar and air pressure were digitized from figures of CWB and may therefore be considered as indicative only. Careful comparison of our model results for $|Q_{Ql}|$ with frequency-independent $\lambda$, with values calculated by CWB show that our model values are about 9% higher.
The reason for this discrepancy is unknown to the author. Still, the shape of variations of IQD certainly follows the same curve, and we are again faced in this simple model with almost the same response as in CBW. An introduction of the variations of friction coefficient with frequency, which is based on the use of quadratic friction in the basic harmonic, obviously does not lead to a better agreement of the model with already mentioned cross-spectral analysis. Unfortunately, the time series of the relevant data used for the cross-spectral analysis were not long enough (thirteen months of data) to provide the frequency resolution necessary for checking abrupt variations in $\lambda$ at low frequency range. Magnitudes of the volume flow through the Strait of Sicily (QS; Fig. 3), and of average elevation of the Western (Z1) and Eastern (Z2) Mediterranean, follow curves which are almost covered for both cases, with $\lambda$ being frequency-dependent, or independent. There are some differences in magnitude of Z1. At frequencies around $7 \times 10^{-4}$ cph (Z1) overshoots the response of inverted barometer by about 16% when $\lambda$ is frequency-independent, and the phase shift at these frequencies is strongly reduced. Otherwise, magnitudes of Z1 and Z2 decrease with frequency, the magnitude of Z1 being smaller than the magnitude of Z2. Phase shifts for frequency-dependent $\lambda$ are the same for both Z1 and Z2, while for frequency-independent, the phase shift of Z2 is larger than the phase shift of Z1 at high frequencies.

There is some scruple about including the friction only at Gibraltar, and not also at Sicily. A two-basin and two-strait model for constant values of linear friction coefficients at both straits ($\lambda_G = \lambda_S = 4.83 \times 10^{-5} \text{ s}^{-1}$) was elaborated by Candela (1991). He found that the only appreciably affected variable when $\lambda_S$ was introduced was the flow through the Strait of Sicily. From Figure 3 it follows that at frequencies below $\sim 4 \times 10^{-3}$ cph the magnitude IQD is much smaller than IQG, therefore the friction coefficient $\lambda_S$ should also be much smaller than $\lambda$ (= $\lambda_G$), and our approximation was reasonable. On the other hand, at higher frequencies it seems that the friction through the Strait of Sicily may have a significant effect also on the flow through the Strait of Gibraltar. The arguments of CBW for dropping the friction term through the Strait of Sicily, based on the geometry ($\lambda_S < \lambda_G$) of this strait, which is much wider than the Strait of Gibraltar, are not too convincing (it seems that wider Sicily strait is also quite shallow, which makes the friction important). Involvement of $\lambda_S$, which should also be proportional to the magnitude of the volume flow $Q_S$, seriously complicates the system of equations, and terms like $\lambda_Q\lambda_S$ appear on the way to the amplitude equations for both volume flows. But although the uniqueness of the solutions should be verified, they would perhaps be in a better agreement with measurements at frequencies above $8 \times 10^{-3}$ cph. This should perhaps be one of the improvements in the model for future work.

Another possibility of improving the model would be the introduction of the Fourier series of the bottom friction stress with higher odd harmonics. But, since the energy lost within one cycle is already well parameterized with the fundamental harmonic, this approach would probably not change results qualitatively.

Although a thorough overview of the work related to the geostrophic control of the flow through the strait and its relatively limited field of application was presented by Pratt (1991), the manner in which the geostrophic control is introduced in the dynamics of straits when friction is important (Garret, 1983; Toulany and Garret, 1984) is still not well understood. On the basis of the last reference CBW claimed that if geostrophic control is present, then the coefficient $\lambda$ should be equal to $fW_Q L_Q$, where $W_Q$ is the width of the Strait of Gibraltar (A$_G$/B$_G$), and $L_Q = 6 \times 10^6 \text{ m}^2$ is its approximate length as mentioned earlier. This means that the frequency-independent part of $\lambda$ is $\sim 3.5 \times 10^{-5} \text{ s}^{-1}$ for the latitude of Gibraltar = 36°N. At the same time, they also point out that geostrophic control may play a role only in straits with $fW_Q L_Q >> \omega^2 \lambda$ (Toulany and Garret, 1984). This last restriction does certainly not hold at frequencies higher than $\sim 4 \times 10^{-3}$ cph at Gibraltar (Fig. 1), but it may be fulfilled below this value. Again, longer time series of data.
are needed to examine the behaviour of the straits at low frequency range. An attempt might be made to combine both mechanisms, the geostrophic control and the linearization of friction which provides the correct energy dissipation within one cycle, by simply allowing $\lambda$ to be the sum of a constant part according to the geostrophic control, together with a frequency-dependent part, which is proportional to the volume flow through the strait. This would actually be tantamount to considering $\lambda$ to be a "generic form of constraint" (CWB). The amplitude equation similar to (2.19) could probably be solved. But, in addition to other serious limitations in the use of geostrophic control (i.e. finite basins in which coastal Kelvin waves may propagate; Wright, 1987), the physical grounds for doing this simple addition do not appear to be strong.

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REFERENCES


